VECTOR CALCULUS WITH LINEAR ALGEBRA
TOPIC I: SETS AND FUNCTIONS

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1. Sets and Elements

A *set* is a collection of *elements*. The elements of a set are sometimes called *members* or *points*. We assume that we can distinguish between different elements, and that we can determine whether or not a given element is in a given set.

The relationship of two elements a and b being the same is *equality* and is denoted $a = b$. The negation of this relation is denoted $a \neq b$, that is, $a \neq b$ means that it is not the case that $a = b$.

The relationship of an element a being a member of a set A is *containment* and is denoted $a \in A$. The negation of this relation is denoted $b \notin A$, that is, $b \notin A$ means that it is not the case that $b \in A$.

A set is determined by the elements it contains. That is, two sets are considered equal if and only if they contain the same elements. We use the symbols "$\Rightarrow$" to mean "implies", and "$\Leftrightarrow$" to mean "if and only if". Then

$$A = B \quad \Leftrightarrow \quad (a \in A \Leftrightarrow a \in B);$$

in English, "A equals B if and only if (a is in A if and only if b is in B)".

We may describe a set by listing its members; such lists are surrounded by braces. For example the set of the first five prime integers is $\{2, 3, 5, 7, 11\}$. If a pattern is clear, we may use dots to indicate an infinite set; for example, to label the set of all prime numbers as P, we may write $P = \{2, 3, 5, 7, 11, 13, \dots\}$. The order of elements in a list is irrelevant in determining a set, for example, $\{5, 3, 7, 11, 2\} = \{2, 3, 5, 7, 11\}$. Also, there is no such thing as the "multiplicity" of an element in a set, for example $\{1, 3, 2, 2, 1\} = \{1, 2, 3\}$.

2. Subsets

If A and B are sets and all of the elements in A are also contained in B, we say that A is a *subset* of B or that A is *contained* in B and write $A \subset B$:

$$A \subset B \quad \Leftrightarrow \quad (a \in A \Rightarrow a \in B);$$

in English, "A is contained in B if and only (a is in A implies a is in B)". Every set is a subset of itself. We say that A is a *proper subset* of B is $A \subset B$ but $A \neq B$.

It follows immediately from the definition of subset that

$$A = B \quad \Leftrightarrow \quad (A \subset B \text{ and } B \subset A);$$

in English, "A equals B if and only if (A is a subset of B and B is a subset of A)."

A set containing no elements is called the *empty set* and is denoted $\varnothing$. Since a set is determined by its elements, there is only one empty set. Note that the empty set is a subset of any set.

Date: August 6, 2017.

3. Set Operations

We may construct new sets as subsets of existing sets by specifying properties. Specifically, we may have a proposition $p(x)$ which is true for some elements x in a set X and not true for others. Then we may construct the set

$$\{x \in X \mid p(x) \text{ is true}\};$$

this is read "the set of x in X such that $p(x)$". The construction of this set is called *specification.* For example, if we let $\mathbb{Z}$ be the set of integers, the set P of all prime numbers could be specified as $P = \{n \in \mathbb{Z} \mid n \text{ is prime}\}$.

Let A and B be subsets of some "universal set" U and define the following set operations:

$$\begin{aligned} \text{Union:} \quad & A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\} \\ \text{Intersection:} \quad & A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\} \\ \text{Complement:} \quad & A \smallsetminus B = \{x \in U \mid x \in A \text{ and } x \notin B\} \end{aligned}$$

The pictures which correspond to these operations are called *Venn diagrams.*

Example 1. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5\}$. Then $A \cap B = \{1, 3, 5\}$, $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$, $A \smallsetminus B = \{7, 9\}$, and $B \smallsetminus A = \{2, 4\}$. □

Example 2. Let A and B be two distinct nonparallel lines in a plane. We may consider A and B as sets of points. Their intersection is a set containing a single point, their union is a set consisting of all points on crossing lines, and the complement of A with respect to B is A minus the point of intersection. □

If $A \cap B = \varnothing$, we say that A and B are *disjoint.*

The following properties are sometimes useful.

- $A = A \cup A = A \cap A$
- $\varnothing \cap A = \varnothing$ and $\varnothing \cup A = A$
- $A \subset B \Leftrightarrow A \cap B = A$
- $A \subset B \Leftrightarrow A \cup B = B$

The following properties state that union and intersection are commutative and associative operations, and that they distribute over each other. These properties are intuitively clear via Venn diagrams.

- $A \cap B = B \cap A$
- $A \cup B = B \cup A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Since $(A \cap B) \cap C = A \cap (B \cap C)$, parentheses are useless and we write $A \cap B \cap C$. This extends to four sets, five sets, and so on. Similar remarks apply to unions.

The following properties of complement are known as *DeMorgan's Laws.* You should draw Venn diagrams of these situations to convince yourself that these properties are true.

- $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$
- $A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$

4. Cartesian Product

Let a and b be elements. The *ordered pair* with first coordinate a and second coordinate b consists of these two elements in the specified order. We denote this ordered pair by (a, b) and declare that it has the following "defining property":

$$(a, b) = (c, d) \quad \Leftrightarrow \quad (a = c \text{ and } b = d).$$

The ordered pair (a, a) is allowed, and $(a, b) = (b, a) \Leftrightarrow a = b$.

The *cartesian product* of the sets A and B is denoted $A \times B$ and is defined to be the set of all ordered pairs whose first coordinate is in A and whose second coordinate is in B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Example 3. Let $A = \{1, 3, 5\}$ and let $B = \{1, 4\}$. Then

$$A \times B = \{(1, 1), (1, 4), (3, 1), (3, 4), (5, 1), (5, 4)\}.$$

In particular, this set contains 6 elements. □

In general, if A contains m elements and B contains n elements, where m and n are natural numbers, then $A \times B$ contains mn elements. Consider the case where $A = B$; then $A \times A$ contains m^2 elements. We sometimes write A^2 to mean $A \times A$.

We have the following properties of cartesian products:

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

5. Numbers

The following familiar sets of numbers have standard names:

Natural Numbers:	$\mathbb{N} = \{0, 1, 2, 3, \dots\}$
Integers:	$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
Rational Numbers:	$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$
Real Numbers:	$\mathbb{R} = \{$ numbers given by decimal expansions $\}$
Complex Numbers:	$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

The following standard notation gives subsets of the real numbers, called *intervals*:

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ (closed)
- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ (open)
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$ (closed)
- $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$ (open)
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$ (closed)
- $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$ (open)

6. Functions

Let A and B be sets. A *function* from a set A to a set B is an assignment of every element in A to a unique element in B. Alternatively, a function is a method of sending each element of A to an element of B.

Let f be a function from A to B. If $a \in A$, the element of B to which a is assigned by f is denoted $f(a)$; in other words, the place in B to which a is sent by f is denoted $f(a)$. We declare that a function must satisfy the following "defining property":

$$\text{for every } a \in A \text{ there exists a unique } b \in B \text{ such that } f(a) = b.$$

If f is a function from A to B, this fact is denoted

$$f : A \to B.$$

We say that f *maps* A *into* B, and that f is a function *on* A. For this reason, functions are sometimes called *maps* or *mappings*. If $f(a) = b$, we say that a is *mapped to* b by f. We may indicate this by writing $a \mapsto b$.

Two functions $f : A \to B$ and $g : A \to B$ are considered *equal* if they act the same way on every element of A:

$$f = g \quad \Leftrightarrow \quad (a \in A \Rightarrow f(a) = g(a)).$$

Thus to show that two functions f and g are equal, select an arbitrary element $a \in A$ and show that $f(a) = g(a)$.

If A is sufficiently small, we may explicitly describe the function by listing the elements of A and where they go; for example, if $A = \{1, 2, 3\}$ and $B = \mathbb{R}$, a perfectly good function is described by $\{1 \mapsto 23.432, 2 \mapsto \pi, 3 \mapsto \sqrt{593}\}$.

However, if A is large, the functions which are easiest to understand are those which are specified by some *rule* or *algorithm*. The common functions of single variable calculus are of this nature.

Example 4. The following can be functions from $\mathbb{R}$ into $\mathbb{R}$:

- $f(x) = 0$;
- $f(x) = x$;
- $f(x) = x^3 + 3x + 17$.

The following can be functions from the set of positive real numbers into $\mathbb{R}$:

- $f(x) = \frac{1}{x}$;
- $f(x) = \sqrt{x}$.

Note that $\frac{1}{x}$ is not a function from $\mathbb{R}$ into $\mathbb{R}$, because it is not defined at $x = 0$. □

Some functions are constructed from existing functions by specifying cases.

Example 5. Let $\mathbb{R}$ be the set of real numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0; \\ x^3 - 1 & \text{if } x \geq 0. \end{cases}$$

Then, for example, $f(-2) = (-2)^2 + 2 = 6$ and $f(2) = 2^3 - 1 = 7$. □

Example 6. Let X be a set and let $A \subset X$. The *characteristic function* of A in X is a function $\chi_A : X \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 0 \text{ if } x \notin A; \\ 1 \text{ if } x \in A. \end{cases}$$

7. Images and Preimages

If $f : A \to B$, the set A is called the *domain* of the function and the set B is called the *codomain.* We often think of a function as taking the domain A and placing it in the codomain B. However, when it does so, we must realize that more than one element of A can be sent to a given element in B, and that there may be some elements in B to which no elements of A are sent.

If $a \in A$, the *image* of a under f is $f(a)$.

If $b \in B$, the *preimage* of b is a subset of A given by

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}.$$

If $C \subset A$, we define the *image* of C under f to be the set

$$f(C) = \{b \in B \mid f(a) = b \text{ for some } a \in A\}.$$

The image of the domain is called the *range* of the function.

If $D \subset B$, we define the *preimage* of D under f to be the set

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}.$$

Notice that $f^{-1}(b)$ is not necessarily a singleton subset of A. For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, then the preimage of the point 4 is

$$f^{-1}(4) = \{2, -2\}.$$

A function $f : A \to B$ is called *surjective* (or *onto*) if

$$\text{for every } b \in B \text{ there exists } a \in A \text{ such that } f(a) = b.$$

Equivalently, f is surjective if $f(A) = B$. This says that every element in B is “hit” by some element from A.

A function $f : A \to B$ is called *injective* (or *one-to-one*) if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

Equivalently, f is injective if for all $b \in B$, $f^{-1}(b)$ contains at most one element in A.

A function $f : A \to B$ is called *bijective* if it is both injective and surjective. Such a function sets up a *correspondence* between the elements of A and the elements of B.

Example 7. First we consider “real-valued functions of a real variable”. This simply means that the domain and the codomain of the function are subsets of $\mathbb{R}$.

- $f(x) = x^3$ is bijective;
- $g(x) = x^2$ is neither injective nor surjective;
- $h(x) = x^3 - 2x^2 - x + 2$ is surjective but not injective;
- $e(x) = 2^x$ is injective but not surjective.

Let $A = \{-1, 1, 2\}$. Some of the images and preimages of A are:

- $f(A) = \{-1, 1, 8\}$;
- $g(A) = \{1, 4\}$;
- $h(A) = \{0\}$;
- $f^{-1}(A) = \{-1, 0, \sqrt[3]{2}\}$;
- $g^{-1}(A) = \{-\sqrt[3]{2}, -1, 1, \sqrt[3]{2}\}$;
- $a^{-1}(A) = \varnothing$.

8. Composition of Functions

Let A, B, and C be sets and let $f : A \to B$ and $g : B \to C$. The *composition* of f and g is the function

$$g \circ f : A \to C$$

given by

$$g \circ f(a) = g(f(a)).$$

The domain of $g \circ f$ is A and the codomain is C. The range of $g \circ f$ is the image under g of the image under f of the domain of f.

Proposition 1. *Let $f : A \to B$ and $g : B \to C$ be surjective functions. Then $g \circ f : A \to C$ is an surjective function.*

Proposition 2. *Let $f : A \to B$ and $g : B \to C$ be injective functions. Then $g \circ f : A \to C$ is an injective function.*

Example 8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x - 9$. Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is given by $g \circ f(x) = x^2 - 9$ and $f \circ g : \mathbb{R} \to \mathbb{R}$ is given by $f \circ g(x) = x^2 - 6x + 9$. □

This example demonstrates that composition of functions is not a commutative operation. However, the next proposition tells us that composition of functions is associative.

Proposition 3. *Let A, B, C, and D be sets and let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.*

9. Restrictions, Identities, and Inverses

Let $f : X \to Y$ be a function and let $Z = f(X)$ be the range of f. The same function f can be viewed as a function $f : X \to Z$. It is standard in this case to call the function, viewed in this way, by the same name. Note that the function $f : X \to Z$ is surjective. Thus any function is a surjective function onto its range.

Let $f : X \to Y$ be a function and let $A \subset X$ be a subset of the domain of f. The *restriction* of f to A is a function

$$f \restriction_A : A \to Y \text{ given by } f \restriction_A (a) = f(a).$$

Thus given any function and any subset of the domain, there is a function which coincides with the original one, but whose domain is the subset. For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ can certainly be viewed as a function on the integers, sending each integer to its square.

Let A be any set. The *identity function* on A if the function $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) = a$ for every $a \in A$. Thus the identity function on A is that function which does nothing to A. The identity function has the property that if $g : A \to C$, then $g \circ \mathrm{id}_A = g$, and if $h : D \to A$, then $\mathrm{id}_A \circ h = h$.

Let $f : A \to B$ be a function. We say that f is *invertible* if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In this case we call g the *inverse* of f. The inverse of a function f is often denoted f^{-1}.

If f is not injective, then f cannot be invertible. Sometimes we restrict the domain of f to a subset on which f is injective to invent a partial inverse.

10. Exercises

Exercise 1. Let $A = \{4, 5, 6, 7, 8, 9, 10, 11\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$, and $C = \{3, 6, 9, 12, 15, 18, 21\}$. Find the indicated set.

(a) $(A \cap B) \setminus C$
(b) $A \setminus (B \cup C)$
(c) $(A \setminus B) \cup C$

Exercise 2. Let A, B, and C be the following subsets of $\mathbb{N}$:

- $A = \{n \in \mathbb{N} \mid n \leq 25\}$;
- $E = \{n \in A \mid n \text{ is even}\}$;
- $O = \{n \in A \mid n \text{ is odd}\}$;
- $P = \{n \in A \mid n \text{ is prime}\}$;
- $S = \{n \in A \mid n \text{ is a square}\}$;

Compute the following sets.

(a) $(P \cup S) \cap O$
(b) $(E \setminus S) \cup P$
(c) $(O \cap S) \times (E \cap S)$

Exercise 3. Let $A = [0, 5]$, $B = (2, 7)$, $C = (6, 9)$, and $D = \{1, 3, 4, 7\}$. Find each of the following sets.

(a) $(A \cup B) \setminus D$
(b) $B \cup (C \cap D)$
(c) $A \setminus D$
(d) $(A \cup C) \setminus D$

Exercise 4. Let $A = \{x \in \mathbb{R} \mid -3 \leq x < 7\}$ and $B = \{x \in \mathbb{R} \mid 1 < x \leq 5\}$.
Find the indicated set.

(a) A
(b) B
(c) $A \cup B$
(d) $A \cap B$
(e) $A \setminus B$

Exercise 5. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5, 7, 9, 11\}$.
Find $C = (A \cup B) \setminus (A \cap B)$.

Exercise 6. Let $D = [2, 10]$ and $E = (\pi, 8]$. Find $F = (D \setminus E) \setminus \mathbb{Z}$.

Exercise 7. Sketch the graph of the set $[1, 3] \times ([1, 4] \setminus [2, 3])$ as a subset of $\mathbb{R}^2$.

Exercise 8. Sketch the graph of the set $([1, 5] \setminus (2, 4)) \times (\{1, 3\} \cup [4, 5])$.

Exercise 9. Let $A = [2, 3) \cup \{4\} \cup (5, 6]$. Sketch the graph of the set $A \times A$.

Exercise 10. Sketch the graph of the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 - 6x + y^2 - 4y \leq 0\}$.

Exercise 11. Draw Venn diagrams which demonstrate the following equations.

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(c) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
(d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Exercise 12. Let A and B be subsets of a set U. The *symmetric difference* of A and B, denoted $A \triangle B$, is the set of points in U which are in either A or B but not in both.

(a) Draw a Venn diagram describing $A \triangle B$.
(b) Find two set expressions which could be used to define $A \triangle B$. These expressions may use A, B, union, intersection, complement, and parentheses,

Exercise 13. Find the domain of the function $f(x) = \frac{\sqrt{x^2-3x-70}}{x^2-64}$. Express your answer in interval notation.

Exercise 14. Find the range of the function $g(x) = x^2 - 4x + 17$. Express your answer in interval notation.

Exercise 15. Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{Z}$ be the integers. Find examples of functions $f : \mathbb{Z} \to \mathbb{N}$ such that:
(a) f is bijective;
(b) f is injective but not surjective;
(c) f is surjective but not injective;
(d) f is neither injective nor surjective.

Exercise 16. Let $\mathbb{N}$ be the set of natural numbers. Let $A = [50, 70] \cap \mathbb{N}$. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = 3n$. Note that A is in both the domain and the codomain of f.
(a) Find the image $f(A)$.
(b) Find the preimage $f^{-1}(A)$.
(c) Is f injective? Is f surjective?

Exercise 17. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - 6x^2 + 11x - 3$. Find $f^{-1}(3)$.

Exercise 18. We would like to define a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by $(p, q) \mapsto \frac{p}{q}$. Unfortunately, this does not make sense. Fix the problem, so that the resulting function is surjective but not injective.

Exercise 19. We would like to define a function $f : \mathbb{Q} \to \mathbb{Z}$ by $\frac{p}{q} \mapsto pq$. Unfortunately, this is not "well-defined". Figure out what this means and fix the problem. Is the resulting function injective?

Exercise 20. Let $f : X \to Y$ be a function and let $A, B \subset X$ and $C, D \subset Y$. Which of the following statements are true? If the statement is false, attempt to construct a counterexample.

(a) $f(A \cup B) \subset f(A) \cup f(B)$
(b) $f(A \cup B) = f(A) \cup f(B)$
(c) $f(A \cap B) \subset f(A) \cap f(B)$
(d) $f(A \cap B) = f(A) \cap f(B)$
(e) $f^{-1}(C \cup D) = f^{-1}(C) \cup f(D)$
(f) $f^{-1}(C \cap D) = f^{-1}(C) \cap f(D)$

Exercise 21. Let $f : X \to Y$ be a function. Which of the following statements are true?

(a) f is surjective if and only if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.
(b) f is injective if and only if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

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